EXISTENCE OF INVARIANT SUBSPACES FOR UNBOUNDED OPERATORS WITH MOMENTS

Abstract: In this text, we adapt some methods of A. Atzmon and G. Godefroy (see [At] and [AtGo]) to prove the existence of invariant subspaces for unbounded operators "with moments" in real Banach spaces. In particular, we extend some results of E. Albrecht and F.-H. Vasilescu (see [AlVa]).

0. PRELIMINARIES.

A well-known result of Scott Brown (see [Br]) asserts that each subnormal operator has a proper invariant subspace. Recently, E. Albrecht and F.-H. Vasilescu have studied the existence of nontrivial (quasi-)invariant subspaces for subnormal families of unbounded operators having sufficiently rich domains, by exploiting the techniques of Thomson and Trent (see [Th] and [Tr] respectively). The aim of this work is to generalize the results of E. Albrecht and F.-H. Vasilescu, in the framework of real Banach spaces, applying the methods of A. Atzmon and G. Godefroy (see [At] and [AtGo]).

We start with some definitions (see [AlVa]).

0.1 Definition: Let $\mathcal{X}$ be a Banach space and let T be a closed unbounded operator with a dense domain $\mathcal{D}(T) \subset \mathcal{X}$. Let $\mathcal{L}$ be a closed subspace of $\mathcal{X}$. We denote by:

$$\mathcal{D}_0(T,\mathcal{L}) = \mathcal{D}(T) \cap \mathcal{L}.$$

The subspace $\mathcal{L}$ is said *invariant under* T if $\mathcal{D}_0(T,\mathcal{L})$ is dense in $\mathcal{L}$ and we have:

$$T(\mathcal{D}_0(T,\mathcal{L})) \subset \mathcal{L}.$$

Similarily, we say that closed linear space $\mathcal{L}$ is *quasi-invariant* under T if $\mathcal{D}(T,\mathcal{L})$ is dense in the subspace $\mathcal{L}$, where $\mathcal{D}(T,\mathcal{L})$ is given by the following equality:

$$\mathcal{D}(T,\mathcal{L}) = \Big\{ x \in \mathcal{D}_0(T,\mathcal{L}); Tx \in \mathcal{L} \Big\}.$$

As mentioned in [AlVa], every invariant subspace under T is quasi-invariant. Moreover, this two notions are identical and coincide with the well-known notion of invariant subspace in the bounded case. We note that there exist quasi-invariant subspaces which are not invariant spaces, see example 3 of [AlVa].

We say that a multi-operator $T = (T_j)_{j\in J}$ (defined in $\mathcal{X}$, J finite subset of $\mathbb{Z}_+$) has an invariant (resp. quasi-invariant) subspace $\mathcal{L}$ if $\mathcal{L}$ is an invariant (resp. quasi-invariant) subspace under each T_j, $j \in J$.

Let T be a densely defined closed operator in the real Banach space $\mathcal{X}$. We denote by $\mathcal{D}^\infty(T)$ the intersection of the domains of all iterates of T.

Following A. Atzmon (see [At]), we say that an operator T is an *operator with* (*Hamburger or Stieltjes*) *moments* in $\mathcal{X}$ if there exists a pair (x,y), $x \in \mathcal{D}^\infty(T)$ and $y \in \mathcal{X}^*$, such that we have the following integral representation:

$$\langle T^n x; y\rangle = \int t^n d\mu(t), \quad \forall n \in \mathbb{Z}_+, \tag{0.1}$$

where μ is a positive measure on $\mathbb{R}$ (respectively $\mathbb{R}^+$) (where the support of μ is not necessarily compact).

For such an operator T, we define following subsets of the space $\mathcal{X}$.

$$H(y,T) = \Big\{x \in \mathcal{D}^\infty(T) / \exists \nu \in \mathcal{M}(\mathbb{R}) \text{ s.t. } \langle T^n x; y\rangle = \int_{\mathbb{R}} t^n d\nu(t), \forall n \in \mathbb{Z}_+\Big\}.$$

$$S(y,T) = \Big\{x \in \mathcal{D}^\infty(T) / \exists \nu \in \mathcal{M}(\mathbb{R}^+) \text{ s.t. } \langle T^n x; y\rangle = \int_0^\infty t^n d\nu(t), \forall n \in \mathbb{Z}_+\Big\}.$$

The sets $H(y,T)$ and $S(y,T)$ be called be the Hamburger set, respectively the Stieltjes set associated to the operator T and to the element $y \in \mathcal{X}^*$.

0.2 Remark: The Hamburger set $H(y,T)$ and the Stieltjes set $S(y,T)$ are nonempty for each $y \in \mathcal{X}^*$ because 0 is included in each one. We can verify easily that they are convex positive cones. Moreover, they are "invariant" under T^2 and T, respectively. Indeed, $\mathcal{D}(T^2)$ and $\mathcal{D}(T)$ both contain $H(y,T)$ and $S(y,T)$, and we have:

$$T^2(H(y,T)) \subset H(y,T) \quad \text{and} \quad T(S(y,T)) \subset S(y,T). \tag{0.2}$$

1. INVARIANT SUBSPACES FOR ONE UNBOUNDED OPERATOR.

In this section, all Banach spaces are assumed to be real, separable and reflexive (as in [At]), if not otherwise specified.

We shall try to adapt the methods of A. Atzmon, who uses measures with compact support. As we study unbounded operators, we are forced to consider measures whose support is not necessarily compact. Some of the needed ingredients already appear in the paper [AtGo].

Let $\overline{H(y,T)}^w$ and $\overline{S(y,T)}^w$ be the closures (in the weak topology) of the Hamburger and Stieltjes sets, respectively, in the Banach space $\mathcal{X}$.

1.1 Lemma: *Let y be a nonnul element in $\mathcal{X}^*$. The sets $\overline{H(y,T)}^w$ and $\overline{S(y,T)}^w$ are strictly included in the Banach space $\mathcal{X}$.*

Proof: As $y \neq 0$, there exists an element $x \in \mathcal{X}$ such that $\langle x,y\rangle = -1$. Assume that $\overline{S(y,T)}^w$ is equal to $\mathcal{X}$. Therefore, there exists a sequence $(z_k)_k$ in $S(y,T)$ which converges weakly to x. In particular, we obtain:

$$0 \leq \mu_k(\mathbb{R}^+) = \langle z_k,y\rangle \rightarrow \langle x,y\rangle = -1,$$

where the $(\mu_k)_k$ are the positive measures associated to the elements $z_k \in S(y,T)$. Hence, $\overline{S(y,T)}^w$ cannot be equal to $\mathcal{X}$.

Similarily, we obtain that $\overline{H(y,T)}^w \neq \mathcal{X}$. ■

1.2 Remark: Assume that there exists a nonnull element $f_0 \in S(y,T)$. Consequently, the boundary $\partial\overline{S(y,T)}^w$ contain a nonnull element too. Let z_0 be one of these elements. Then, we choose $z_1 \in \mathcal{X}\backslash\overline{S(y,T)}^w$ such that we have:

$$0 < ||z_1 - z_0|| < \frac{||z_0||}{3}. \tag{1.1}$$

In particular, we have $z_1 \neq 0$.

As $S(y,T)$ is a convex set, $\overline{S(y,T)}^w$ is a convex positive closed cone included in the Banach space $\mathcal{X}$, which is reflexive. For this reason there exists a vector $u \in \overline{S(y,T)}^w$ such that distance between $\overline{S(y,T)}^w$ and z_1 can be written as

$$d(z_1,\overline{S(y,T)}^w) = d(z_1,u). \tag{1.2}$$

An easy calculation shows that this vector is nonnul.
Obviously, we can make the same construction for the Hamburger set $H(y,T)$.

1.3 Proposition : *Let T be an unbounded operator with dense domain in $\mathcal{X}$. Let also $y \in \cap_{i=1}^{\infty}\mathcal{D}(T^{*i}) \subset \mathcal{X}^*, y \neq 0$. Then the positive convex cone $S(y,T)$ contains $\mathcal{D}^\infty(T) \cap \partial\overline{S(y,T)}^w$.*

Proof : Fix $u \in \overline{S(y,T)}^w$. There exists a sequence $(x_p)_p$ in $S(y,T)$ such that $\langle x_p,\phi\rangle \to \langle u,\phi\rangle, \quad \forall\phi \in \mathcal{X}^*$. Hence, there exists a sequence of Stieltjes measures $(\nu_p)_p$ with the properties :

$$\langle T^n x_p,y\rangle = \int_0^\infty t^n d\nu_p(t), \quad \forall n,p \geq 0. \tag{1.3}$$

It is well known that $\gamma = (\gamma_i)_{i\geq 0}$ is a Stieltjes moments sequence if and only if we have :

$$\sum_{i,j\geq 0} a_i\overline{a_j}\gamma_{i+j} \geq 0 \quad \text{and} \quad \sum_{i,j\geq 0} a_i\overline{a_j}\gamma_{i+j+1} \geq 0, \tag{1.4}$$

for every finite family in $\mathbb{C}^{\mathbb{N}}$.

As $x_p \in S(y,T)$ for all $p \geq 0$, by the remark above we obtain :

$$\sum_{i,j\geq 0} a_i\overline{a_j}\langle T^{i+j}x_p,y\rangle \geq 0 \quad \text{and} \quad \sum_{i,j\geq 0} a_i\overline{a_j}\langle T^{i+j+1}x_p,y\rangle \geq 0. \tag{1.5}$$

Moreover, as we have assumed that $y \in \cap_{i=1}^{\infty}\mathcal{D}(T^{*i})$, the following linear functional is continuous for all pairs of indices (i,j) :

$$\phi(*) = \langle T^{i+j} *,y\rangle = \langle *,T^{*(i+j)}y\rangle \in \mathcal{X}^*, \tag{1.6}$$

Consequently, we obtain the following convergence :

$$\langle T^a x_p,y\rangle = \langle x_p,T^{*a}y\rangle \to \langle u,T^{*a}y\rangle = \langle T^a u,y\rangle, \tag{1.7}$$

for every positive integer a, because we have assumed that $u \in \mathcal{D}^\infty(T)$. As in the criterion for the Stieltjes moments, the sums are finite, and passing to the limit, we obtain :

$$\sum_{i,j\geq 0} a_i\overline{a_j}\langle T^{i+j}u,y\rangle \geq 0 \quad \text{and} \quad \sum_{i,j\geq 0} a_i\overline{a_j}\langle T^{i+j+1}u,y\rangle \geq 0. \tag{1.8}$$

This shows that $(\langle T^a u,y\rangle)_a$ is a Stieltjes moments sequence, which allows us to conclude that $u \in S(y,T)$. ∎

Before giving a first generalization of Atzmon's result, we need some technical lemmas, concerning polynomials of degree two (see also [AtGo]).

Let $j \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$ satisfying $2j - k \leq 0$. Then the function f defined by

$$f(s,t) = 1 + s(1+t)^j + s^2(1+t)^k \tag{1.9}$$

is nonnegative on the half plane $\mathbb{R} \times \mathbb{R}^+$, as one can easily see.

1.4 Lemma : *Let u be a nonnull vector in $S(y,T)$. Then for all $s \in \mathbb{R}$, and for every pair $(j,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ satisfying the condition $2j - k \leq 0$, the elements $u + s(I+T)^j u + s^2(I+T)^k u$ are included in the positive cone $S(y,T)$.*

Proof : As u is included in $S(y,T)$, it is included in particular in $\mathcal{D}^\infty(T)$, and there exists a positive measure μ with support included in $\mathbb{R}^+$, such that :

$$\langle T^n u, y \rangle = \int_0^{+\infty} t^n d\mu(t), \quad \forall n \geq 0. \tag{1.10}$$

It is obvious that $u + s(I+T)^j u + s^2(I+T)^k u$ is a element of the linear space $\mathcal{D}^\infty(T)$, which satisfies :

(1.11)

$$\langle T^n(u + s(I+T)^j u + s^2(I+T)^k u), y \rangle = \int_0^{+\infty} t^n f(s,t) d\mu(t) = \int_0^{+\infty} t^n d\mu_{j,k,s}(t),$$

where the positive measure $\mu_{j,k,s}$ is given by :

$$d\mu_{j,k,s}(t) = \Big(1 + s(1+t)^j + s^2(1+t)^k\Big) d\mu(t). \tag{1.12}$$

This measure is effectively positive, due to the preceding remark on the function $f(s,t)$, which is nonnegative on the half plane $\mathbb{R} \times \mathbb{R}^+$. ■

We now follow some ideas from [Si] and [At]. Take u an element of $\mathcal{D}^\infty(T)$ and $v \in \mathcal{X}$ arbitrary. We define the functions $F_{j,k}$ from $\mathbb{R}$ to $\mathbb{R}^+$, for every pair $(j,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$, by the formula

$$F_{j,k}(s) = ||v - s(I+T)^j u - s^2(I+T)^k u||. \tag{1.13}$$

1.5 Proposition : *The functions $F_{j,k}$, $(j,k) \in \mathbb{Z}^2_+$ are differentiable at 0, and we have the equality :*

$$F'_{j,k}(0) = -\langle (I+T)^j u, \Phi_0 \rangle,$$

where Φ_0 is the unique linear functional of $\mathcal{X}^*$ with $||\Phi_0|| = 1$, satisfying $\langle v, \Phi_0 \rangle = ||v||$.

Proof : One can follow the lines of the proof given by G. Köthe (see [Ko], pp 347-350) where a similar proposition is proved (only for polynomials of degree one). We omit the details. ■

For every family $(\eta_\alpha)_{\alpha \in A}$ of vectors, we denote by $Vect(\eta_\alpha)_{\alpha \in A}$ the linear space spanned by this family. If it does not follow from the context, we will specify if it is a real or a complex linear space.

1.6 Lemma : *Let $\mathcal{X}$ be a Banach space (real or complex) and let T be an unbounded operator in $\mathcal{X}$. If there exists a linear functional $\phi \in \mathcal{X}^*$ such that $\langle (I+T)^j u, \phi \rangle = 0$ for every $j \geq 0$, with $u \in \mathcal{D}^\infty(T)$, then ϕ is in $Vect\{T^j u\}^\perp$.*

Proof: The assertion is obtained by an easy induction. ■

Theorem 1.7 below uses the notation from Remark 1.2.

1.7 Theorem: *Let T be a closed operator with dense domain $\mathcal{D}(T)$ in $\mathcal{X}$. Assume that T is an operator with Stieltjes moments for a pair of vectors $(x,y) \in \mathcal{D}^\infty(T) \times \mathcal{D}^\infty(T^*)$. Moreover, assume that the distance $d(z_1,\overline{S(y,T)}^w)$ is attained at an element $u \in \mathcal{D}^\infty(T)$. Then T has a nontrivial quasi-invariant subspace.*

Proof: Using Proposition 1.3 and Remark 1.2, the element u is nonnul and included in $S(y,T)$. We can apply the Lemma 1.4 to the vector u to obtain that all the vectors $u+s(I+T)^j u+s^2(I+T)^k u$ are in the cone $S(y,T)$, $(j,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ satisfying the condition $2j-k \leq 0$. We define the functions $F_{j,k}$ on $\mathbb{R}$ letting:

$$F_{j,k}(s) = ||(z_1 - u) - s(I+T)^j u - s^2(I+T)^k u||. \tag{1.14}$$

As $u + s(I+T)^j u + s^2(I+T)^k u$ is in $S(y,T)$, we have:

$$F_{j,k}(s) \geq d(z_1,\overline{S(y,T)}^w) = d(z_1,u) = F_{j,k}(0). \tag{1.15}$$

Hence, the functions $F_{j,k}$ have a global minimum in 0. Moreover, using Proposition 1.5, these functions are differentiable and:

$$F'_{j,k}(0) = -\langle (I+T)^j u,\Phi_0\rangle, \tag{1.16}$$

where Φ_0 is the unique linear functional of norm 1 in $\mathcal{X}^*$ satisfying $\langle z_1 - u,\Phi_0\rangle = ||z_1 - u||$. Therefore, we have the equalities:

$$\langle (I+T)^j u,\Phi_0\rangle = 0, \quad \forall j \geq 0. \tag{1.17}$$

Finally, we use Lemma 1.6 to conclude that:

$$\langle T^j u,\Phi_0\rangle = 0, \quad \forall j \geq 0. \tag{1.18}$$

Let $Y = \overline{Vect\{T^j u, \quad j \geq 0\}}$. The space Y is nontrivial because it contains $u \neq 0$ and is different from $\mathcal{X}$ because there exists a nonnull linear functional Φ_0 in its orthogonal. Moreover, we have:

$$\mathcal{D}(T,Y) = \big\{x \in \mathcal{D}(T) \cap Y \text{ such that } Tx \in Y\big\} \supset Vect\big\{T^j u, \quad j \geq 0\big\}.$$

Hence, Y is a nontrivial quasi-invariant subspace. ■

If we translate the preceding Theorem 1.7 in the bounded case, we obtain the following Corollary:

1.8 Corollary: *Let T be a bounded operator defined on $\mathcal{X}$. Assume that T is an operator with Stieltjes moments for a pair of vectors $(x,y) \in \mathcal{X} \times \mathcal{X}^*$. Then T has a nontrivial invariant subspace.*

Proof: If T is bounded, its domain $\mathcal{D}(T) = \mathcal{X}$. Moreover, the distance from a closed convex set to a point is attained at a vector of $\mathcal{X}$ which is in the domain of T. So, the space Y of the preceding theorem is invariant (in the usual sense for bounded operators), using the continuity of the operator T. ■

1.9 Remark: If we are in the bounded case, the measure is necessarily with compact support included in $[-\rho(T),\rho(T)]$, (where $\rho(T)$ is the spectral radius of T) as it is noticed in [AtGo]. So the previous corollary is not an extension of their result, only a version of it.

2- COMPLEMENT ON THE UNBOUNDED CASE.

In Theorem 1.7, we have assumed that the distance between the closed convex positive cone, which is always attained, is attained at an element $u \in \mathcal{D}^\infty(T)$. We begin with a proposition which allows us to remove this condition. As a matter of fact, we just obtain conditions on the given operator T, and moreover, which imply the existence of invariant subspaces rather than of quasi-invariant ones.

2.1 Proposition : *Let T be a closed operator with dense invariant domain $\mathcal{D}(T) \subset \mathcal{X}$. Assume that T is an operator with Stieltjes moments for a pair $(x,y) \in \mathcal{D}^\infty(T) \times \mathcal{D}^\infty(T^*)$ and that the space $Vect(T^{*n}y)_{n\geq 0}$ is dense in $\mathcal{X}^*$. Then each element $u \in \overline{S(y,T)}^w$ is in $\mathcal{D}^\infty(T)$.*

Proof: Let $(x_m)_{m\geq 0}$ be a sequence of vectors in $S(y,T) \subset \mathcal{D}^\infty(T)$ weakly-convergent to $u \in \overline{S(y,T)}^w$. So there exists a sequence $(z_m)_m$ included in $Vect\{x_m,m \geq 0\}$ which converges strongly to u. This sequence is a subset of $\mathcal{D}^\infty(T)$ due to the properties on $(x_m)_{m\geq 0}$.

We begin to prove that the sequence $(Tz_m)_{m\geq 0}$ is a Cauchy one in $\mathcal{X}$. Indeed, we have (taking a strictly positive ε) :

$$\begin{aligned} ||Tz_n - Tz_m|| &= \sup_{\phi\in\mathcal{X}^*,||\phi||\leq 1} \langle Tz_n - Tz_m,\phi\rangle \\ &\leq \tfrac{\varepsilon}{2} + |\langle Tz_n,p(T^*)y\rangle - \langle Tz_m,p(T^*)y\rangle|, \end{aligned}$$

where p is a polynomial in one variable. Moreover, we know that :

$$\langle Tz_m,p(T^*)y\rangle \to \langle u,T^*p(T^*)y\rangle.$$

Therefore, there exists an integer N_0 such that, for all $m,n \geq N_0$, we have :

$$|\langle Tz_n,p(T^*)y\rangle - \langle Tz_m,p(T^*)y\rangle| \leq \frac{\varepsilon}{2}.$$

As we are in a Banach space, the Cauchy sequence converges to an element u'. Consequently, the couple (z_m,Tz_m) converges in the graph $\mathcal{G}(T)$ of T to (u,u'). Because we have assumed that T is a closed operator, we obtain that :

$$u \in \mathcal{D}(T) \quad \text{and} \quad Tu = u'.$$

Now, we have $u' \in \overline{S(y,T)}^w$ because $(Tz_m)_{m\geq 0}$ is a weakly-convergent sequence to u'. We can verify this fact on the dense family $(T^{*n}y)_{n\geq 0}$ of $\mathcal{X}^*$. We repeat the argument above one more time (applied to u') to obtain that :

$$(u',Tu') = (u',u'') \in \mathcal{G}(T), \quad \text{which means} \quad u \in \mathcal{D}(T^2).$$

And, step by step, we conclude that the element u is in $\mathcal{D}^\infty(T)$. ∎

2.2 Theorem : *Let T be a closed operator with a dense invariant domain $\mathcal{D}(T) \subset \mathcal{X}$. Assume that T is an operator with Stieltjes moments for a couple*

$(x,y) \in \mathcal{D}^\infty(T) \times \mathcal{D}^\infty(T^*)$ *and that the space* $Vect(T^{*n}y)_{n\geq 0}$ *is dense in* $\mathcal{X}^*$. *Then* T *has a nontrivial invariant subspace in* $\mathcal{X}$.

Proof: We proceed as in the proof of Theorem 1.7. We denote by u a element which minimalizes the distance between z_1 and the positive convex cone $S(y,T)$. Using the preceding Proposition 2.1, we know that u is in the linear space $\mathcal{D}^\infty(T)$. Then we can apply the Theorem 1.7. So $Y = \overline{Vect\{T^j u, \quad j \geq 0\}}$ is a nontrivial quasi-invariant subspace. We have to show that Y is also an invariant subspace *i.e.*:

$$T\Big(Y \cap \mathcal{D}(T)\Big) \subset Y.$$

Let $z \in Y \cap \mathcal{D}(T)$: two cases may happen. First, if z can be written as $p(T)u$, we have:

$$Tz = Tp(T)u \in Y.$$

In the other case, using the density of $Vect\big\{T^j u, \quad j \geq 0\big\}$, there exists a sequence of polynomials $(p_n)_{n\geq 0}$ such that:

$$p_n(T)u \to z.$$

As in the previous proposition, we will show that $(Tp_n(T)u)_n$ is a convergent sequence. For $\varepsilon > 0$, we have:

$$\begin{aligned} ||Tp_n(T)u - Tp_m(T)u|| &= \sup_{\phi \in \mathcal{X}^*, ||\phi|| \leq 1} \langle Tp_n(T)u - Tp_m(T)u, \phi\rangle \\ &= \sup_{\phi \in \mathcal{X}^*, ||\phi|| \leq 1} \left(\langle Tp_n(T)u,\phi\rangle - \langle Tp_m(T)u,\phi\rangle\right) \\ &\leq \tfrac{\varepsilon}{2} + |\langle Tp_n(T)u, q(T^*)y\rangle - \langle Tp_m(T)u, q(T^*)y\rangle|, \end{aligned}$$

for a polynomial q in one variable. Moreover, we have $\langle Tp_n(T)u, q(T^*)y\rangle \to \langle z, T^*q(T^*)y\rangle$, which implies that there exists N_0 such that $\forall n,m \geq N_0$ we have:

$$||Tp_n(T)u - Tp_m(T)u|| \leq \varepsilon.$$

Consequently, $(Tp_n(T)u)_n$ is a Cauchy sequence in $\mathcal{X}$. Let w be the limit in this Banach space. Therefore, we obtain that:

$$\mathcal{G}(T) \ni (p_n(T)u, Tp_n(T)u) \to (z,w),$$

where $\mathcal{G}(T)$ is the graph of the operator T. As $\mathcal{G}(T)$ is closed, we obtain that $w = T(z)$. For any integer n, $Tp_n(T)u \in Y$, which is a closed subset of $\mathcal{X}$. Hence $T(z) = w$ is also in the linear space Y.

Hence, Y is a closed nontrivial invariant subspace for the closed operator T. ■

2.3 Corollary: *Let* T *be a closed operator with a dense invariant domain* $\mathcal{D}(T) \subset \mathcal{X}$. *Assume that* T *is an operator with Stieltjes moments for a pair* $(x,y) \in \mathcal{D}^\infty(T) \times \mathcal{D}^\infty(T^*)$. *Then either* T *or* T^* *has a nontrivial invariant subspace in* $\mathcal{X}$ *or* $\mathcal{X}^*$.

Proof: It is a straightforward consequence of the preceding Theorem 2.2. Indeed, if the family of linear functionals $(T^{*n}y)_{n\geq 0}$ is dense in $\mathcal{X}^*$ then using result 2.2, we obtain that T has a nontrivial invariant subspace in $\mathcal{X}$. Else, $(T^{*n}y)_{n\geq 0}$ is not dense then this linear space of functionals is a nontrivial quasi-invariant subspace for T^* in $\mathcal{X}^*$. Let $\mathcal{L}$ be the closure of $Vect\big\{T^{*n}y, \quad n \geq 0\big\}$. As T is closed with dense domain, $\mathcal{D}(T^*)$ is also dense and $T^{**} = T$. So we can use the same argument as in the proof of Theorem 2.2 (we can assume that $Vect\big\{T^n x, \quad n \geq 0\big\}$ is dense in $\mathcal{X}$). ■

Application to real Hilbert spaces : Let $\mathcal{X}$ be a real Hilbert space (obviously reflexive), and let T be an unbounded self-adjoint (and so closed) operator with dense domain $\mathcal{D}(T)$ which is invariant under T. Then we can use the functional calculus of T which exists even if we are in the real case (see for example Chapter VI of [Cr]). We obtain that T is an operator with Hamburger moments for each couple $(x,x) \in \mathcal{D}^\infty(T) \times \mathcal{D}^\infty(T)$. Indeed, we have :

$$\langle T^n x, x\rangle = \int_{\mathbb{R}} t^n d\langle E(t)x, x\rangle, \quad \forall x \in \mathcal{D}^\infty(T),$$

where E is the spectral measure associated to the self-adjoint operator T. Assuming $T \neq 0$, we have either $E(\mathbb{R}^+) \neq 0$ or $E(\mathbb{R}^-) \neq 0$. If we assume that $E(\mathbb{R}^+) \neq 0$, then we obtain for a chosen x :

$$\langle T^n E(\mathbb{R}^+)x, E(\mathbb{R}^+)x\rangle = \int_{\mathbb{R}^+} t^n d\langle E(t)x_+, x_+\rangle, \quad x_+ = E(\mathbb{R}^+)x \in \mathcal{D}^\infty(T).$$

If $E(\mathbb{R}^-) \neq 0$, we may replace T by $-T$ and reduce the problem to the previous one. In both cases, we deduce that T has a nontrivial invariant subspace. Consequently, we obtain the following proposition.

2.4 Proposition : *Let H be a real Hilbert space and let T be a self-adjoint operator with invariant domain. Then T has a nontrivial invariant subspace.*

We have dealt with the self-adjoint case, the natural following step is to approach the subnormal operators. In the bounded case, we know that every subnormal operator on a Hilbert space has a nontrivial invariant subspace, see [Br]. We will show that, under natural conditions, this result remains true in the unbounded case (if we are in a real Hilbert space). For definitions and criterions of subnormal unbounded operators, see for example [StSz]. If we are in the complex case, E. Albretch and F.-H. Vasilescu have shown that these operators have nontrivial quasi-invariant subspaces (see Theorem 11 of [AlVa]).

2.5 Theorem : *Let H be a real Hilbert space and T be a subnormal operator with invariant domain $\mathcal{D}(T) \subset H$. Then T has a nontrivial invariant subspace.*

Proof : Let N be a normal extension of T, defined in a real Hilbert space $\mathcal{K} \supset \mathcal{H}$. For each vector $x \in \mathcal{D}^\infty(T)$, we obtain that

$$\langle T^n x_+, x_+\rangle = \langle N^n x_+, x_+\rangle = \int_{\mathbb{R}} t^n d\langle E(t)x_+, x_+\rangle, \quad x_+ = E(\mathbb{R}^+)x \in \mathcal{D}^\infty(T),$$

where E is the spectral measure associated to the normal operator N. So, using Corollary 2.3, there exists a closed nontrivial subspace $\mathcal{M}$ such that we have :

$$T\Big(\mathcal{D}(T) \cap \mathcal{M}\Big) \subset \mathcal{M} \quad \text{ or } \quad T^*\Big(\mathcal{D}(T^*) \cap \mathcal{M}\Big) \subset \mathcal{M}.$$

If the first case holds, we have proved the theorem. If not, for all $y \in \mathcal{M}^{\perp} \cap \mathcal{D}(T)$ we have :

$$\langle Ty,x \rangle = \langle y,T^* x \rangle = 0, \quad \forall x \in \mathcal{D}(T^*).$$

So we have $Ty \in \mathcal{M}^{\perp}$. Consequently, in this case $\mathcal{M}^{\perp}$ is an invariant subspace (obviously nontrivial because $\mathcal{M}$ is non trivial) under the operator T. ■

3. THE MULTI-OPERATORIAL CASE.

Most of the results valid for one operator can be adapted to multi-operators. The proofs are similar. We just have to assume that the couples (x,y) are now in the set $\Big(\mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l)\Big) \times \Big(\mathcal{D}^\infty(T_1^*) \cap \cdots \cap \mathcal{D}^\infty(T_l^*)\Big)$. We only have to change Proposition 1.3 because the criterion used to have a Stieltjes moment sequence is valable for $n = 1$.

Let $T = (T_1, \cdots ,T_l)$ be a family of unbounded operators. We denote by $S(y,T_1, \cdots ,T_l)$ the following set :

$$\Big\{ x \in \mathcal{D}^\infty(T)/\exists \nu \in \mathcal{M}(\mathbb{R}^l) \text{ such that } \langle T^\alpha x ; y \rangle = \int t^\alpha d\nu(t),\ \forall \alpha \in \mathbb{Z}_+^l \Big\}.$$

We replace Proposition 1.3 by the following one :

3.1 Proposition : *Assume that* $y \in \Big(\mathcal{D}^\infty(T_1^*) \cap \cdots \cap \mathcal{D}^\infty(T_l^*)\Big)$. *Then the positive convex cone* $S(y,T_1, \cdots ,T_l)$ *is included in the set* $\mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l) \cap \partial \overline{S(y,T_1, \cdots ,T_l)}^w$.

Proof: Let $u \in \overline{S(y,T_1, \cdots ,T_l)}^w$. There exists a sequence $(x_p)_p$ such that : $\langle x_p,\phi \rangle \to \langle u,\phi \rangle, \quad \forall \phi \in \mathcal{X}^*$. This sequence $(x_p)_p$ satisfies the property $x_p \in S(y,T_1, \cdots ,T_l) \subset \mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l)$. So, there exists a sequence of Stieltjes measures $(\nu_p)_p$ satisfying :

$$\langle T^\alpha x_p,y \rangle = \int_{\mathbb{R}_+^l} t^\alpha d\nu_p(t), \quad \forall \alpha \in \mathbb{Z}_+^l, \quad \forall p \geq 0.$$

Let $u \in \overline{S(y,T_1, \cdots ,T_l)}^w$. There exists a sequence $(x_p)_p$ such that : $\langle x_p,\phi \rangle \to \langle u,\phi \rangle, \quad \forall \phi \in \mathcal{X}^*$.

A sequence $\gamma = (\gamma_\alpha)_{\alpha \geq 0}$ is a multi-sequence of Stieltjes moments if and only if the linear functional associated L_γ is nonnegative for all polynomials q nonnegative on $\mathbb{R}_+^l$ (see [Ha] and [Ha2]). If we denote by $\gamma_\alpha = \langle T^\alpha u,y \rangle$ and if $q(t) = \sum_\beta c_\beta t^\beta$, we obtain :

$$\begin{aligned} L_\gamma(q) \quad &= \sum_{\beta \geq 0} c_\beta \gamma_\beta = \sum_{\beta \geq 0} c_\beta \langle T^\beta u,y \rangle = \sum_{\beta \geq 0} c_\beta \langle u,T^{*\beta} y \rangle \\ &= \lim_{p \to \infty} \sum_{\beta \geq 0} c_\beta \langle x_p,T^{*\beta} y \rangle = \lim_{p \to \infty} \int_{\mathbb{R}_+^l} q(t) d\nu_p(t) \geq 0. \end{aligned}$$

Therefore, $\gamma = (\gamma_\alpha)_{\alpha \geq 0}$ is a multi-sequence of Stieltjes moments, which allows us to conclude that $u \in S(y,T_1, \cdots ,T_l)$. ■

With the same techniques, we obtain the multi-variable results :

3.2 Proposition : *Let $T = (T_1, \cdots, T_l)$ be an unbounded multi-operator in $\mathcal{X}$, consisting of closed operators. We assume that there exists a dense linear space $\mathcal{D}$ included in $\mathcal{D}(T_1) \cap \cdots \cap \mathcal{D}(T_l)$, invariant under each T_i $(i = 1, \cdots, l)$, and that the multi-operator $T = (T_1, \cdots, T_l)$ has Stieltjes moments for a couple $(x,y) \in \Big(\mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l)\Big) \times \Big(\mathcal{D}^\infty(T_1^*) \cap \cdots \cap \mathcal{D}^\infty(T_l^*)\Big)$. We also assume that the linear functionals $(T^{*\alpha}y)_\alpha$ are dense in $\mathcal{X}^*$. Then every element $u \in \overline{S(y,T_1, \cdots, T_l)}^w$ belongs to $\mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l)$.*

3.3 Theorem : *Let $T = (T_1, \cdots, T_l)$ be an unbounded multi-operator in $\mathcal{X}$, consisting of closed operators. We assume that there exists a dense linear space $\mathcal{D}$ included in $\mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l)$ and invariant under each T_i $(i = 1, \cdots, l)$. Assume that the multi-operator $T = (T_1, \cdots, T_l)$ has Stieltjes moments for a couple $(x,y) \in \Big(\mathcal{D}^\infty(T_1) \cap \cdots \cap \mathcal{D}^\infty(T_l)\Big) \times \Big(\mathcal{D}^\infty(T_1^*) \cap \cdots \cap \mathcal{D}^\infty(T_l^*)\Big)$. At last, we assume that the family of linear functionals $(T^{*\alpha}y)_\alpha$ is dense in $\mathcal{X}^*$. Then, the multi-operator T has a nontrivial invariant subspace.*

3.4 Remarks : As in the preceding part, if we have a family of self-adjoint operators $(A_1, \cdots, A_l)$ in a real Hilbert space $\mathcal{H}$ (with $\mathcal{D}^\infty(A_1, \cdots, A_l)$ a dense subspace of $\mathcal{H}$), then there exists a joint nontrivial invariant subspace under the operators A_i $(i = 1, \cdots, l)$.

We can obtain a result similar to Theorem 2.5 in the multi-variable case, whose proof is also similar.

3.6 Example : Let $\mathcal{H}$ be the real Hilbert space $\mathcal{L}^2(\mathbb{R}^m)$. We use the orthogonal basis (the Chebyshev-Hermite functions) :

$$f_P(x_1, \cdots, x_m) = e^{(x_1^2 + \cdots + x_m^2)/2} \frac{\partial^{|P|}}{\partial x_1^{p_1} \cdots \partial x_m^{p_m}} \Big(e^{-(x_1^2 + \cdots + x_m^2)}\Big), \quad P \in \mathbb{Z}_+^m.$$

We define the operators of creation A_j $(j = 1, \cdots, m)$ by :

$$\begin{cases} A_j(*) = (x_j - \dfrac{\partial}{\partial x_j})/\sqrt{2} \\ \\ \mathcal{D}(A_j) = Vect\{f_P, \quad P \geq 0\}. \end{cases}$$

We can prove, using [De] or [De2], that $A = (A_1, \cdots, A_m)$ is a commuting subnormal multi-operator (see example of [De2]), satisfying :

$$A_j(f_P) = \frac{-1}{\sqrt{2}} f_{P+e_j}, \quad j = 1, \cdots, m,$$

where $(e_j)_j$ is the canonical basis of $\mathbb{R}^m$. Therefore, by the previous remark, this multi-operator has an invariant nontrivial subspace.

We can prove this property also by finding directly the elements x and y in Theorem 3.4. Take for example $x = f_0$ and $y = \sum b_P f_P$. Then we have :

$$\langle A^P f_0, y \rangle = (-1)^{|P|} b_P \sqrt{\pi}^m \sqrt{2}^{|P|} P!.$$

So, if we chose for every $P \geq 0$:

$$b_P = \Big((-\sqrt{2})^{|P|} \sqrt{\pi}^m (P + (1, \cdots, 1))!\Big)^{-1},$$

we obtain first that $y \in \mathcal{H}$, because:

$$||b_P f_P|| \leq C \prod_{j=1}^{m} j^{-j/2},$$

where C is a positive constant. Moreover, we have the following equalities (for every $P \in \mathbb{Z}_+^m$):

$$\langle A^P f_0, y \rangle = \frac{1}{(p_1+1)\cdots(p_m+1)} = \int_{[0,1]^m} t^P d\lambda(t),$$

where $d\lambda(t)$ is the Lebesgue measure on the compact $[0,1]^m$. Therefore, the Stieltjes set $H(y,(A_1,\cdots,A_m))$ is nonempty. So using the multi-variable version of Theorem 2.5, we obtain that the multi-operator A has a non trivial invariant subset.

4. APPLICATION FOR COMPLEX HILBERT SPACES.

Our goal in this section is to extend results of E. Albrecht and F.-H. Vasilescu on the existence of quasi-invariant subspaces (in special cases), proving a complex version. A natural way to do this is to decompose each operator into the real and a imaginary part.

4.1 Definition: Let T be a densely defined operator in a Hilbert space $\mathcal{H}$. We say that T has a *Cartesian decomposition* if there exist two symmetric operators T_1 and T_2 satisfying $\mathcal{D}(T_1) = \mathcal{D}(T_2)$ and $T = T_1 + iT_2$ (see for example [Ot] for some comments on this definition).

We can easily see that a densely defined operator has a Cartesian decomposition if and only if its domain satisfies the inclusion $\mathcal{D}(T) \subset \mathcal{D}(T^*)$. And then we obtain the relations:

$$T_1 = \frac{T+T^*}{2} \quad \text{and} \quad T_2 = \frac{T-T^*}{2i}.$$

For example, every densely defined subnormal T operator has a Cartesian decomposition. Indeed, we have for all $(x,y) \in \mathcal{D}(T) \times \mathcal{D}(T)$:

$$\phi(x) = \langle Tx,y \rangle = \langle Nx,y \rangle = \langle x,N^*y \rangle,$$

where N is a normal extension of T, because $y \in \mathcal{D}(T) \subset \mathcal{D}(N) = \mathcal{D}(N^*)$. So, ϕ is a linear and continuous functional on the domain $\mathcal{D}(T)$, which means that $y \in \mathcal{D}(T^*)$. So we have $\mathcal{D}(T) \subset \mathcal{D}(T^*)$.

If N is a normal operator, in the Cartesian decomposition we obtain two self-adjoint operators A_1 and A_2. We define the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ included in $\mathcal{H}$, associated to the orthonormal basis $(e_k)_k$, letting:

$$\mathcal{H}_{\mathbb{R}} = \Big\{ x \in \mathcal{H} / \langle x,e_k \rangle \in \mathbb{R}, \quad \forall k \geq 0 \Big\}.$$

Obviously, $\mathcal{H}_{\mathbb{R}}$ is a real Hilbert space with the scalar product of $\mathcal{H}$. Even if the operators A_1 and A_2 are self-adjoint, the real space is not necessarily invariant under these two operators. But if this property is true, then we can define their restrictions to $\mathcal{H}_{\mathbb{R}}$, which are operators (denoted by B_1 and B_2) in $\mathcal{H}_{\mathbb{R}}$. Then, we obtain for all $x \in \mathcal{H}_{\mathbb{R}}$:

$$\langle B^\alpha x,x\rangle = \langle A^\alpha x,x\rangle = \int_{\mathbb{R}^2} t^\alpha d\langle E(t)x,x\rangle = \int_{\mathbb{R}^2} t^\alpha d\mu_x(t).$$

Up to a transformation of A_i in $-A_i$ (as in the Remark 3.5), we obtain that the pair $A = (A_1,A_2)$ has a commun nontrivial invariant subspace M (a real one). Letting $\mathcal{F} = M + iM$, we obtain a nontrivial invariant subspace under N :

$$N(\mathcal{F}) = A_1(\mathcal{F}) + iA_2(\mathcal{F}) \subset \mathcal{F}.$$

The fact that normal operators have invariant subspaces can be easily seen using the spectral measure of the operators. But the previous method can be used for subnormal operators using the Cartesian decomposition.

We will say that an operator T, in a complex Hilbert space $\mathcal{H}$, has the property $(\mathcal{R})$ if there exists an orthonormal basis $(e_i)_{i\geq 0}$ of $\mathcal{H}$ such that :

$$\langle Te_i,e_j\rangle \in \mathbb{R}, \quad \forall (i,j) \in \mathbb{Z}_+^2.$$

For example, if $\mathcal{L}^2(\mathbb{R}^m)$ is the complex Hilbert space, the creation multi-operator A (of the example 3.6) has the property $(\mathcal{R})$.

4.2 Theorem : *Let $(T_1,\cdots,T_l)$ be a family of subnormal multi-operators in a Hilbert space $\mathcal{H}$ (of dimension bigger than 1) with the property $(\mathcal{R})$. Assume that there exists a linear subspace $\mathcal{D} \subset \mathcal{D}(T_1) \cap \cdots \cap \mathcal{D}(T_l)$ included in $\mathcal{H}$ such that we have $T_j(\mathcal{D}) \subset \mathcal{D}$, for all $j = 1,\cdots,l$. Then the family $(T_1,\cdots,T_l)$ has a nontrivial invariant subspace.*

Proof: Note first that the subnormality condition implies that $T_jT_kx = T_kT_jx$ for all $x \in \mathcal{D}$ and for all pair $(j,k) \in \{1,\cdots,l\}^2$. So we can take without ambiguity the powers of $(T_1,\cdots,T_l)$ on $\mathcal{D}$. We can assume that $\mathcal{D}$ is dense, otherwise the closure of this space is a nontrivial invariant subspace. We use the Cartesian decomposition of each T_j letting $T_j = S_{1,j} + iS_{2,j}$. We can define the restrictions of each $S_{k,j}$ to the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ because these operators are symmetric. And using the functional calculus of a normal extension $(N_j = A_{1,j} + iA_{2,j})_{j\in J}$ we obtain for all $x \in \mathcal{H}_{\mathbb{R}} \cap \mathcal{D}$:

$$\langle S^\alpha x,x\rangle = \langle A^\alpha x,x\rangle = \int_{\mathbb{R}^{2l}} t^\alpha d\langle E(t)x,x\rangle = \int_{\mathbb{R}^{2l}} t^\alpha d\mu_x(t).$$

With the same argument as before, we obtain a Stieltjes moments sequence for the family of unbounded operators $S = (S_{k,j})_{\{1,2\}\times\{1,\cdots,l\}}$ (where we have denote by A the family $(A_{k,j})_{\{1,2\}\times\{1,\cdots,l\}}$). Hence, we conclude as above. ■

4.3 Remark : As we have noticed at the beginning, every invariant subspace is also a quasi-invariant one. Therefore the preceding Theorem 4.2 can be seen as a generalization of the Theorem 11 in [AlVa], where the authors have proved the existence of nontrivial quasi-invariant subspaces, provided we assume the condition $(\mathcal{R})$.

5. ON THE ARENS ALGEBRAS $L^\omega(\mu)$.

Let Ω be a locally compact Hausdorff space and let μ be a positive finite Borel measure on Ω. We denote by $L^\omega(\mu) = \cap_{p\geq 1} L^p(\mu)$ as in [Ar] and [AlVa]. Associated with the family $(||.||_p)_{p\geq 1}$, we obtain a metrizable locally convex topological vector space. Moreover (see [FlWl]), the dual space may be identified with :

$$L^{1+}(\mu) = \bigcup_{p>1} L^p(\mu).$$

The algebra $L^\omega(\mu)$ is reflexive satisfying for all $p > 1$:

$$L^\infty(\mu) \subset L^\omega(\mu) \subset L^p(\mu) \subset L^{1+}(\mu) \subset L^1(\mu).$$

We recall some well-known fact on these sets. Let $\mathcal{A}$ un a subalgebra of $L^\omega(\mu)$, of dimension bigger than 2, containing the constants. We define by $\mathcal{A}^p(\mu)$ the closure of $\mathcal{A}$ in $L^p(\mu)$. Similarily, we denote by $\mathcal{A}^\omega(\mu)$ the closure of $\mathcal{A}$ in $L^\omega(\mu)$ which is a subalgebra of $L^\omega(\mu)$ satisfying :

$$\mathcal{A}^\omega(\mu) = \bigcap_{p\geq 1} \mathcal{A}^p(\mu)$$

For every $a \in \mathcal{A}^\omega(\mu)$, we denote the operators S_a and N_a by :

$$S_a \left\{ \begin{array}{ccc} \mathcal{D}(S_a) & \to & \mathcal{A}^2(\mu) \\ f & \to & af \end{array} \right. \qquad N_a \left\{ \begin{array}{ccc} \mathcal{D}(N_a) & \to & L^2(\mu) \\ f & \to & af \end{array} \right.$$

where $\mathcal{D}(S_a)$ and $\mathcal{D}(N_a)$ are respectively the domain of S_a and N_a given by $\{f \in \mathcal{A}^2(\mu); af \in \mathcal{A}^2(\mu)\}$ and $\{f \in L^2(\mu); af \in L^2(\mu)\}$. So the family $\{S_a, a \in \mathcal{A}^\omega(\mu)\}$ is a subnormal family with normal extension $\{N_a, a \in \mathcal{A}^\omega(\mu)\}$. In [AlVa], it is proved that this family has a nontrivial quasi-invariant subspace, see Theorem 9. Using our methods, we can prove a stronger result in the real case. Indeed, this family has a nontrivial invariant subspace :

Theorem : *Let $\mathcal{A}$ be a subalgebra of the real Arens algebra $L^\omega(\mu)$ having dimension bigger than* 2*. Then the multiplication operators S_a, $a \in \mathcal{A}^\omega(\mu)$, have a proper invariant subspace in $\mathcal{A}^2(\mu)$.*

If we are in the complex case, we have to assume that the multiplication operators satisfy the property $\mathcal{R}$ for an orthonormal basis of $L^2(\mu)$.

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