Growth of coefficients of universal Dirichlet series

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Abstract. In this work, we study universal Dirichlet series introduced by F. Bayart [3] and [4]. In particular we obtain estimation on growth of coefficients. We can then compare several classes of universal Dirichlet series.

1 Introduction

Let $f(s) = \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series and let $\sigma_a(f)$ be its abscissa of absolute convergence, defined by

$$\sigma_a(f) = \inf\left\{\sigma \in \mathbb{R}\ ;\ \sum_{n\geq 1} |a_n| n^{-\sigma} \text{ converges}\right\}.$$

We define also the abscissa of convergence $\sigma(f) = \inf\{\sigma \in \mathbb{R}\ ;\ \sum_{n\geq 1} a_n n^{-\sigma}$ converges $\}$. We denote the p-th partial sum $S_p(f) = \sum_{n=1}^{p} a_n n^{-s}$. Let $\mathbb{C}_+$ be the half-plane of complex numbers with strictly positive real part. We denote by $\mathcal{D}_a(\mathbb{C}_+)$ the set of Dirichlet series which are absolutely convergent on $\mathbb{C}_+$. This space $\mathcal{D}_a(\mathbb{C}_+)$, endowed with the topology given by the following family of semi-norms

$$\left\|\sum_{n\geq 1} a_n n^{-s}\right\|_\sigma = \sum_{n\geq 1} |a_n| n^{-\sigma} \quad (\sigma > 0),$$

is a Fréchet space. In the following, we fix $\tilde{\sigma} = (\sigma_k)_{k\geq 0}$ to be a strictly decreasing sequence of real numbers which converges to 0. Then, the distance associated to the Fréchet space is defined by, for f and g in $\mathcal{D}_a(\mathbb{C}_+)$,

$$d_{\tilde{\sigma}}(f,g) = \sum_{n\geq 0} \frac{1}{2^n} \frac{||f-g||_{\sigma_n}}{1 + ||f-g||_{\sigma_n}}.$$

Definition 1.1 *Let K be a compact set included in $\mathbb{C}$. This set is admissible for Dirichlet series if $\mathbb{C} \setminus K$ is connected, and if we can obtain the following representation $K = K_1 \cup \ldots \cup K_d$, where the K_i should be contained in disconnected strips $S_i = \{z \in \mathbb{C}\ ;\ a_i \leq \Re(z) \leq b_i\}$ with breadth strictly less than $1/2$ ($b_i - a_i < 1/2$).*

We denote by $\mathbb{C}_-$ the left half plane $\{s \in \mathbb{C}\ ;\ \Re(s) < 0\}$. We can now express the version of Mergelyan's theorem for Dirichlet series included in $\mathcal{D}_a(\mathbb{C}_+)$.

Theorem 1.2 **[3]** *Let $K \subset \overline{\mathbb{C}_-}$ be an admissible compact set for Dirichlet series, let f be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$ and let g be a continuous function*

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on K which is analytic in $\overset{\circ}{K}$. For every couple of fixed positive real numbers σ and ε, there exists a Dirichlet polynomial h satisfying

$$\sup_{z \in K} |h(z) - g(z)| < \varepsilon \text{ and } ||h - f||_\sigma < \varepsilon.$$

We define also the following sets $\mathcal{W}_a$, $\mathcal{W}_d$ and $\mathcal{W}_1$ of universal Dirichlet series from $\mathcal{D}_a(\mathbb{C}_+)$. The sets $\mathcal{W}_a$ and $\mathcal{W}_d$ have been yet introduced in [3] and [9] respectively. We recall them.

Definition 1.3 *We denote by $\mathcal{W}_a$ the set of all Dirichlet series $h \in \mathcal{D}_a(\mathbb{C}_+)$ satisfying : for every admissible compact set $K \subset \overline{\mathbb{C}_-}$, for every function g, continuous on K and analytic in $\overset{\circ}{K}$, there exists a sequence of integers $(\lambda_n)_{n\geq 0}$ such that we have*

$$\sup_{z \in K} |S_{\lambda_n}(h)(z) - g(z)| \underset{n \to +\infty}{\longrightarrow} 0.$$

Definition 1.4 *We denote by $\mathcal{W}_d$ the set of all Dirichlet series $h \in \mathcal{D}_a(\mathbb{C}_+)$ satisfying : for every admissible compact set $K \subset \overline{\mathbb{C}_-}$, for every Dirichlet series f in $\mathcal{D}_a(\mathbb{C}_+)$ without constant term and for every function g, continuous on K and analytic in $\overset{\circ}{K}$, there exists a sequence of integers $(\lambda_n)_{n\geq 0}$ such that we have*

$$\begin{cases} \sup_{z \in K} |S_{\lambda_n}(h)(z) - g(z)| \underset{n \to +\infty}{\longrightarrow} 0 \\ \\ h^{(\lambda_n)} \underset{n \to +\infty}{\longrightarrow} f \text{ in } \mathcal{D}_a(\mathbb{C}_+). \end{cases}$$

Clearly we have the following inclusion $\mathcal{W}_d \subset \mathcal{W}_a$. It is well-known that $\mathcal{W}_d$ and $\mathcal{W}_a$ are G_δ-dense sets (see [3] and [9]).

Definition 1.5 *We denote by $\mathcal{W}_1$ the set of all Dirichlet series $h \in \mathcal{D}_a(\mathbb{C}_+)$ satisfying : for every admissible compact set $K \subset \mathbb{C}_-$, for every function g, continuous on K and analytic in $\overset{\circ}{K}$, there exists a sequence of integers $(\lambda_n)_{n\geq 0}$ such that we have*

$$\sup_{z \in K} |S_{\lambda_n}(h)(z) - g(z)| \underset{n \to +\infty}{\longrightarrow} 0.$$

The set $\mathcal{W}_1$ differs from $\mathcal{W}_a$ by the fact that the intersection of the compact sets K with the imaginary axis must be an empty set. Obviously using similar methods $\mathcal{W}_1$ is also a G_δ-dense set and we have the following inclusions $\mathcal{W}_d \subset \mathcal{W}_a \subset \mathcal{W}_1$. These three sets are analogous of the set of universal Taylor series defined in [8], [13], [12] respectively. For survey and similar results, we can also refer to [10]. As in the analytic case [12], we obtain first estimates on the growth of coefficients of universal Dirichlet series (in the sense of $\mathcal{W}_a$). It is Theorem 2.2. In the second hand we prove a decomposition theorem with estimates on the coefficients for all series of $\mathcal{D}_a(\mathbb{C}_+)$.

Theorem *Let $f = \sum_{n\geq 1} d_n n^{-s}$ be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$. Then, there exist $g_1 = \sum_{n\geq 1} a_n n^{-s}$ and $g_2 = \sum_{n\geq 1} b_n n^{-s}$ in $\mathcal{W}_1$ such that $f = g_1 + g_2$ on $\mathbb{C}_+$ with the condition*

$$\limsup_{n \in \mathbb{N}^*} n|a_n| = \limsup_{n \in \mathbb{N}^*} n|b_n| = \limsup_{n \in \mathbb{N}^*} n|d_n|.$$

This theorem is a Dirichlet version of theorem 5.1 from [12]. As a consequence, we deduce the strict inclusion between $\mathcal{W}_1$ and $\mathcal{W}_a$.
At last, in the universal set $\mathcal{W}_1$, a natural question is to know whether some of universal Dirichlet series converge and moreover are continuous on the imaginary axis. This property is true for analytic function on the unit disk, see [12]. To prove this, A. Melas, V. Nestoridis and I. Papadoperakis study universality on the Banach space $A(\mathbb{D})$ of analytic function on $\mathbb{D}$, continuous on the torus $\mathbb{T}$. To obtain such result for universal Dirichlet series in the section 4, we have choosen the point of view of the Wiener-Dirichlet algebra. We prove then the existence of universal Dirichlet series which are continuous on the imaginary axis.

2 Some properties of $\mathcal{W}_a$ and $\mathcal{W}_d$.

In this section, we study the growth of coefficients of universal Dirichlet series in $\mathcal{W}_a$ or $\mathcal{W}_d$. Note that such series converge nowhere on the imaginary axis. Taking as K a singleton $\{it_0\}$, $t_0 \in \mathbb{R}$, and two different values, we see that series diverge at every point it_0. Hence its abscissas of convergence and absolute convergence are both exactly equal to 0. We obtain a more precise result on the asymptotic behaviour of the universal coefficients.

Lemma 2.1 *Let* $\sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series in* $\mathcal{W}_a$. *Let* $(\varepsilon_n)_{n\in\mathbb{N}}$ *be a decreasing sequence such that*

$$\sum_{n=2}^{\infty} \frac{\varepsilon_n}{n\log(n)} < +\infty.$$

Then, we have

$$\sum_{n=2}^{\infty} \frac{|a_n|}{e^{\sqrt{\varepsilon_n \log(n)}}} = +\infty.$$

Proof. Let be $\delta_n = e\varepsilon_n$ for all integers. There exists n_0 such that $\sum_{n=n_0}^{\infty} \frac{\delta_n}{n\log(n)} < \frac{1}{2}$. We define the functions from $i\mathbb{R}$ which are $2i\pi$-periodic letting

$$\begin{cases} H_n(it) = \dfrac{n\log(n)}{\delta_n}\pi & \text{for } |t| < \dfrac{\delta_n}{n\log(n)}, \\ H_n(it) = 0 & \text{for } \dfrac{\delta_n}{n\log(n)} \leq |t| \leq \pi. \end{cases}$$

We put $\hat{f}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(it)m^{it}dt$. An easy calculation gives

$$\hat{H}_n(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_n(it)dt = 1$$

$$\hat{H}_n(m) = \frac{\sin\left(\dfrac{\delta_n}{n\log(n)}\log(m)\right)}{\dfrac{\delta_n}{n\log(n)}\log(m)}, \quad m \neq 1.$$

Let $N \geq n_0$ be an integer, we can approximate the Dirichlet polynomial $1 + \sum_{m=1}^{N-1} a_m m^{-s}$ by a subsequence of partial sums of f uniformly on the compact set $\{it; t \in [-\frac{1}{2}, \frac{1}{2}]\}$. Therefore there exists an integer $M > N$ such that $\left|1 - \sum_{m=N}^{M} a_m m^{-it}\right| < \dfrac{1}{2}$ for all $t \in [-\frac{1}{2}, \frac{1}{2}]$. Hence we have ($\Re$ means the real part)

$$\frac{1}{2} \leq \Re\left(\sum_{m=N}^{M} a_m m^{-it}\right). \tag{1}$$

We define the convolution product $f(it) = H_{n_0} * \cdots * H_M(it)$, where

$$h * g(it) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(ix)g(it - ix)dx.$$

Note that f is a non-negative $2i\pi$ periodic function satisfying $f(it) = 0$ for $\frac{1}{2} < t \leq \pi$. Hence, multiplying both members of (1) by $f(it)$ and using an integration, we obtain

$$\frac{1}{2} \leq \sum_{m=N}^{M} \Re\left(a_m \int_{-\pi}^{\pi} f(it)m^{-it}dt\right).$$

Using the parity of f and triangle inequality we have

$$\frac{1}{2} \leq \sum_{m=N}^{M} |a_m| \times |\hat{f}(m)|.$$

Moreover, we can calculate $\hat{f}(m)$

$$\hat{f}(m) = \prod_{n=n_0}^{M} \frac{\sin\left(\dfrac{\delta_n}{n\log(n)}\log(m)\right)}{\dfrac{\delta_n}{n\log(n)}\log(m)}.$$

As $(\delta_n)_{n\in\mathbb{N}}$ is a decreasing sequence and the series $\sum_{n\geq 2} \dfrac{\delta_n}{n\log(n)}$ converges, we must have $\lim_{n\to+\infty} \delta_n = 0$. Therefore, there exists an integer N such that we have the following two inequalities $\dfrac{\delta_{n_0}}{n_0\log(n_0)}\log(N) > e$ and $\delta_N < e$. For every $m \in \{N, \ldots, M\}$, we have

$$\frac{\delta_{n_0}}{n_0\log(n_0)}\log(m) \geq \frac{\delta_{n_0}}{n_0\log(n_0)}\log(N) > e \text{ and } \frac{\delta_m}{m\log(m)}\log(m) < \delta_m \leq \delta_N < e.$$

Then there exists an integer $k \in \{n_0, \cdots, m-1\}$ such that

$$\frac{\delta_k}{k\log(k)}\log(m) \geq e \quad \text{and} \quad \frac{\delta_{k+1}}{(k+1)\log(k+1)}\log(m) < e.$$

We obtain also, because the sequence δ_n is decreasing,

$$|\hat{f}(m)| \leq \prod_{n=n_0}^{k} \frac{n\log(n)}{\delta_n \log(m)} \leq \Big(\frac{k\log(k)}{\delta_k \log(m)}\Big)^{k+1-n_0} \leq \Big(\frac{1}{e}\Big)^{k+1-n_0}.$$

Moreover $(k+1)^2 \geq (k+1)\log(k+1) > \dfrac{\delta_{k+1}}{e}\log(m) \geq \varepsilon_m \log(m)$ implies $k+1 > \sqrt{\varepsilon_m \log(m)}$. We obtain

$$\sum_{m=N}^{M} |a_m| \frac{e^{n_0}}{e^{\sqrt{\varepsilon_m \log(m)}}} \geq \frac{1}{2}.$$

Since this holds for infinitely many pairs (N, M), we have the conclusion. $\square$

Theorem 2.2 *Let* $\sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series in* $\mathcal{W}_a$*. Let* $(b_n)_{n\in\mathbb{N}}$ *be a decreasing sequence such that* $\sum_{n=2}^{\infty} \dfrac{b_n}{n\log(n)} < +\infty$*. Then, we have*

$$\limsup_{n\in\mathbb{N}^*} \frac{n|a_n|}{e^{\sqrt{b_n \log(n)}}} = +\infty.$$

Proof. Assume that there exists a real number M such that $|a_n| \leq \dfrac{M}{n} e^{\sqrt{b_n \log(n)}}$ for all $n \geq 1$ integer. Let w_n be $\max(b_n, \dfrac{1}{\sqrt{\log(n)}})$ and let be $\varepsilon_n = (\sqrt{w_n} + \sqrt{b_n})^2$. Note that the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is decreasing. Moreover the series $\sum_{n\geq 2} \dfrac{\varepsilon_n}{n\log(n)}$ converges due to the hypothesis on the sequence $(b_n)_{n\in\mathbb{N}}$ and the Bertrand criterion. So, applying Lemma 2.1, we obtain

$$\sum_{n=1}^{\infty} \frac{|a_n|}{e^{\sqrt{\varepsilon_n \log(n)}}} = +\infty. \tag{2}$$

But we have for every positive integer A

$$\sum_{n=1}^{A} \frac{|a_n|}{e^{\sqrt{\varepsilon_n \log(n)}}} \leq M \sum_{n=1}^{A} \frac{1}{n} \frac{1}{e^{\sqrt{\varepsilon_n \log(n)} - \sqrt{b_n \log(n)}}} = M \sum_{n=1}^{A} \frac{1}{n} \frac{1}{e^{\sqrt{w_n \log(n)}}}.$$

Using inequality $w_n \geq 1/\sqrt{\log(n)}$, we deduce

$$\sum_{n=1}^{A} \frac{|a_n|}{e^{\sqrt{\varepsilon_n \log(n)}}} \leq M \sum_{n=1}^{A} \frac{1}{n} \frac{1}{e^{\sqrt{\sqrt{\log(n)}}}}.$$

But the series $\sum_{n\geq 1} \frac{1}{n} \frac{1}{e^{\sqrt{\sqrt{\log(n)}}}}$ converges which contradicts the equality (2). As a consequence, we have

$$\limsup_{n\in\mathbb{N}^*} \frac{n|a_n|}{e^{\sqrt{b_n \log(n)}}} = +\infty.$$

$\square$

Corollary 2.3 *Let* $\sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series in* $\mathcal{W}_a$*. Then, we have, for every integer* k*,*

$$\limsup_{n\in\mathbb{N}^*} \frac{n|a_n|}{(\log(n))^k} = +\infty.$$

Proof. Let $(b_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$\forall n \geq 2,\ b_n = \frac{k^2(\log(\log(n)))^2}{\log(n)}.$$

This sequence is decreasing (for n sufficiently large) and the series $\sum_{n=2}^{\infty} \frac{b_n}{n\log(n)}$ converges. Using Theorem 2.2, we deduce that

$$\limsup_{n\geq 2} \frac{n|a_n|}{e^{\sqrt{k^2(\log(\log(n)))^2}}} = \limsup_{n\geq 2} \frac{n|a_n|}{\log(n)^k} = +\infty$$

$\square$

Remark 2.4 For the inverse inequality, we know by construction that it is possible to build universal Dirichlet series $\sum_{j\geq 1} a_j j^{-s}$ satisfying $a_j = o(j^{-r})$, for any $r < 1$ (see [3]). Hence, contrarily to the analytic case [12], $\mathcal{W}_a \cap \mathcal{H}^2 \neq \emptyset$ where $\mathcal{H}^2$ is the analogous of $H^2(\mathbb{D})$ for Dirichlet series, see for instance [2].

Moreover using ideas of J.P. Kahane, we have the following proposition (see [13] Proposition 3.2).

Proposition 2.5 *Let* f *be a Dirichlet series in* $\mathcal{D}_a(\mathbb{C}_+)$*. Then, there exist* g_1 *and* g_2 *in* $\mathcal{W}_d$ *such that* $f = g_1 + g_2$*.*

Proof. We use a translation homeomorphism $T_f : \mathcal{D}_a(\mathbb{C}_+) \to \mathcal{D}_a(\mathbb{C}_+)$ defined by $T_f(h) = f - h$. As $\mathcal{W}_d$ is a dense G_δ set, its image by T_f is also a dense G_δ set. Using Baire's theorem, $\mathcal{W}_d \cap [f - \mathcal{W}_d]$ is no void. Let g_1 be an element of this intersection. It follows that $g_1 \in \mathcal{W}_d$ and $g_1 = f - g_2$ for an element $g_2 \in \mathcal{W}_d$. $\square$

As for the set $\mathcal{W}_a$ [3], there exists a relation between the set $\mathcal{W}_d$ and the notion of universality in the sense of Menchoff. We donot repeat the proof, it suffices to follow Proposition 3.1 of [13].

Proposition 2.6 *Let be* $S = \sum_{n\geq 1} a_n n^{-s} \in \mathcal{W}_d$ *and let* g *and* h *be two measurable functions from* $\mathbb{R}$ *into* $[-\infty; +\infty]$*. Then there exists a subsequence* S_{k_m} *of the partial sums of* S *such that*

$$\Re(S_{k_m}(it)) \to g(t) \quad \textit{and} \quad \Im(S_{k_m}(it)) \to h(t),$$

almost everywhere on $\mathbb{R}$.

3 Some properties of $\mathcal{W}_1$.

We recall the definition of this set.

Definition 3.1 *We denote by* $\mathcal{W}_1$ *the set of all Dirichlet series* $h \in \mathcal{D}_a(\mathbb{C}_+)$ *satisfying : for every admissible compact set* $K \subset \mathbb{C}_-$*, for every function* g*, continuous on* K *and analytic in* $\overset{\circ}{K}$*, there exists a sequence of integers* $(\lambda_n)_{n\geq 0}$ *such that*

$$\sup_{z\in K} |S_{\lambda_n}(h)(z) - g(z)| \underset{n\to+\infty}{\longrightarrow} 0.$$

As mentionned in the introduction, $\mathcal{W}_1$ is a G_δ-dense set in $\mathcal{D}_a(\mathbb{C}_+)$. The proof is similar of the case $\mathcal{W}_d$ (see [9] or remark on theorem 6 [3]). Moreover note that such series have abscissa of convergence and absolute convergence exactly equal to 0. To see that, it suffices to take as K a singleton $\{z_0\}$, with $\Re(z_0) < 0$. Nevertheless what happens on imaginary axis ?

Notation 3.2 In the following, we denote the set of Dirichlet polynomials with coefficients in $\mathbb{Q}+i\mathbb{Q}$ by the sequence $(f_j)_{j\in\mathbb{N}}$. Moreover there exists a sequence of admissible compact sets K_ρ satisfying for each admissible (for Dirichlet series) compact set $K \subset \overline{\mathbb{C}_-}$, there exists a nonnegative integer ρ_0 such that $K \subset K_{\rho_0}$ [9]. We denote also by $||.||_\rho$ the supremum norm on K_ρ.

At last we put for Dirichlet polynomial $P(s) = \sum_{n=1}^{n_0} a_n n^{-s}$ the degree $\deg(P) = n_0$ $(a_{n_0} \neq 0)$.

The proposition 2.5 is also true for the sets $\mathcal{W}_1$. Here we give another version of this proposition with some more conditions on the growth of coefficients. Before this, we need a more precise version of Mergelyan's Theorem for Dirichlet series.

Lemma 3.3 *Let* K *be admissible for Dirichlet series compact set included in* $\mathbb{C}_-$*. Let also* g *be a continuous function on* K *and analytic in the interior of* K*. For every pair* ε, σ *of strictly positive real numbers, there exists a Dirichlet polynomial* $h(s) = \sum_{n\geq 1} h_n n^{-s}$ *satisfying*

$$\begin{cases} \sup_{z\in K} |h(z) - g(z)| < \varepsilon \\ ||h||_\sigma < \varepsilon \\ n|h_n| < \varepsilon. \end{cases}$$

Proof. We use the notations of Lemma 2 from [3] with the special case $f = 0 \in \mathcal{D}_a(\mathbb{C}_+)$. We write $K = K_1 \cup \cdots \cup K_d$, then there exists a d-tuple of positive real numbers $\sigma_1 < \cdots < \sigma_d$ such that the following Dirichlet polynomial h

$$h(s) = \sum_{l=1}^{d} \sum_{j=n_l+1}^{m_l} b_j^{(l)} j^{-\sigma_l} j^{-s} = \sum_{j=n_1+1}^{\infty} h_j j^{-s}$$

satisfies $\sup_{z \in K} |h(z) - g(z)| < \varepsilon$ and $||h||_\sigma < \varepsilon$. The choice of n_1 is arbitrary (due to [1]). Moreover, from result of [1], the modulus of the complex numbers $b_j^{(l)}$ are upper bounded by 1. Therefore, we obtain for all $j \in \mathbb{N}^*$

$$|jh_j| \leq j|b_j^{(l)} j^{-\sigma_l}| \leq j^{-\sigma_1+1} \leq n_1^{-\sigma_1+1}.$$

We just have to choose an integer n_1 satisfying $n_1^{-\sigma_1+1} < \varepsilon$ to complete the proof which is possible because the compact set doesnot intersect the imaginary axis. We can use a translation of $\sigma_1 > 1$ such that the first part K_1 of the compact set satisfies $\sigma_1 + K_1 \subset \{s \in \mathbb{C}; \frac{1}{2} < \Re(s) < 1\}$. $\square$

Remark 3.4 Note that we use the condition $K \subset \mathbb{C}_-$ to obtain a control on the $n|h_n|$ appearing in $h(s)$ (from the previous lemma), which isnot possible in the $\mathcal{W}_a$ and $\mathcal{W}_d$ cases.

Corollary 3.5 *Let K be an admissible, for Dirichlet series, compact set included in $\mathbb{C}_-$. Let also g be a continuous function on K and analytic in the interior of K. For every pair ε, σ of strictly positive real numbers and for every strictly positive integer λ, there exists a Dirichlet polynomial $h(s) = \sum_{n \geq 1} h_n n^{-s}$ satisfying*

$$\begin{cases} \sup_{s \in K} |g(s) - \lambda^{-s} h(s)| < \varepsilon \\ ||\lambda^{-s} h(s)||_\sigma < \varepsilon \\ n\lambda |h_n| < \varepsilon. \end{cases}$$

Proof. Using the notations of Lemma 3.3, for every ε_1 there exists a Dirichlet polynomial h such that $\sup_{s \in K} |g(s)\lambda^s - h(s)| < \varepsilon_1$, $||h||_\sigma < \varepsilon_1$ and $n|h_n| < \varepsilon_1$. Therefore, we have

$$\left(\inf_{s \in K} |\lambda^s| \right) \left(\sup_{s \in K} |g(s) - \lambda^{-s} h(s)| \right) < \varepsilon_1.$$

We just have to choose ε_1 such that we have $\max \left(\frac{\varepsilon_1}{\inf_{s \in K} |\lambda^s|}; \lambda \varepsilon_1 \right) < \varepsilon$. $\square$

We can now use main ideas from [12] to obtain the following result.

Theorem 3.6 *Let $f = \sum_{n \geq 1} d_n n^{-s}$ be a Dirichlet series in $\mathcal{D}_a(\mathbb{C}_+)$. There exist $g_1 = \sum_{n \geq 1} a_n n^{-s}$ and $g_2 = \sum_{n \geq 1} b_n n^{-s}$ in $\mathcal{W}_1$ such that $f = g_1 + g_2$ on $\mathbb{C}_+$ with the condition*

$$\limsup_{n \in \mathbb{N}} n|a_n| = \limsup_{n \in \mathbb{N}} n|b_n| = \limsup_{n \in \mathbb{N}} n|d_n|.$$

Proof. First, we study the case $\limsup_{n\in\mathbb{N}} n|d_n| = +\infty$. Using Proposition 2.5, there exist g_1 and g_2 in $\mathcal{W}_a \subset \mathcal{W}_1$ satisfying $f = g_1 + g_2$. The conclusion is given by Corollary 2.3.

Case $\limsup_{n\in\mathbb{N}} n|d_n| < +\infty$: we have a countable family of pair (K_{ρ_i}, f_{j_i}). Let be $\lambda_1 = 1$, then using Corollary 3.5, there exists Dirichlet polynomial $P_1(s) = \sum_{n\geq 1} p_{1,n} n^{-s}$ such that $||f_{j_1}(s) - \lambda_1^{-s} P_1(s)||_{\rho_1} < 1$, $||\lambda_1^{-s} P_1(s)||_{\sigma_1} < 1$ and $n\lambda_1 |p_{1,n}| < 1$ (for every $n \in \mathbb{N}$). Let be $\mu_1 > \lambda_1 + \deg(P_1) \geq \lambda_1$ such that $\limsup_{n\in\mathbb{N}} n|d_n| - \max\{l|d_l|; \mu_1 > l > \lambda_1 + \deg(P_1)\} < 1$. Then, we define the Dirichlet polynomial R_1 letting

$$R_1(s) = \sum_{n=\lambda_1}^{\mu_1 - 1} d_n n^{-s} - \lambda_1^{-s} P_1(s).$$

Using Corollary 3.5, there exists a Dirichlet polynomial $Q_1(s) = \sum_{n\geq 1} q_{1,n} n^{-s}$ satisfying $n\mu_1 |q_{1,n}| < 1$ (for every $n \in \mathbb{N}$),

$$||Q_1||_{\sigma_1} < 1 \quad \text{and} \quad ||f_{j_1}(s) - R_1(s) - \mu_1^{-s} Q_1(s)||_{\rho_1} < 1.$$

Let λ_2 be integer satisfying $\lambda_2 > \mu_1 + \deg(Q_1) \geq \mu_1 > \lambda_1$ and $\limsup_{n\in\mathbb{N}} n|d_n| - \max\{l|d_l|; \lambda_2 > l > \mu_1 + \deg(Q_1)\} < 1$ and we define Dirichlet polynomial F_1

$$F_1(s) = \sum_{n=\mu_1}^{\lambda_2 - 1} d_n n^{-s} - \mu_1^{-s} Q_1(s).$$

We construct step by step the sequences $\tilde{\lambda} = (\lambda_k)_{k\geq 1}$ and $\tilde{\mu} = (\mu_k)_{k\geq 1}$. Assume that we have

$$1 = \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_{k-1} < \mu_{k-1} < \lambda_k$$

and that the polynomials P_i, Q_i, R_i and F_i are constructed for $i = 1, \ldots, k-1$. Using Corollary 3.5, there exists a Dirichlet polynomial $P_k(s) = \sum_{n\geq 1} p_{k,n} n^{-s}$ such that $||\lambda_k^{-s} P_k(s)||_{\sigma_k} \leq \frac{1}{k^2}$, $n\lambda_k |p_{k,n}| < \frac{1}{k^2}$ (for every $n \in \mathbb{N}$) and

$$||f_{j_k}(s) - \sum_{j=1}^{k-1} \left(\lambda_j^{-s} P_j(s) + F_j(s)\right) - \lambda_k^{-s} P_k(s)||_{\rho_k} < \frac{1}{k^2}.$$

Let be $\mu_k > \lambda_k + \deg(P_k)$ such that

$$\limsup_{n\in\mathbb{N}} n|d_n| - \max\{l|d_l|; \mu_k > l > \lambda_k + \deg(P_k)\} < \frac{1}{k^2}$$

and let R_k be

$$R_k(s) = \sum_{n=\lambda_k}^{\mu_k - 1} d_n n^{-s} - \lambda_k^{-s} R_k(s).$$

Then, by Corollary 3.5, there exists a Dirichlet polynomial $Q_k(s) = \sum_{n\geq 1} q_{k,n} n^{-s}$ satisfying $||Q_k||_{\sigma_k} < \frac{1}{k^2}$, $n\mu_k |q_{k,n}| < \frac{1}{k^2}$ (for every $n \in \mathbb{N}$) and

$$||f_{j_k}(s) - \sum_{j=1}^{k-1} \left(\mu_j^{-s} Q_j(s) + R_j(s)\right) - R_k(s) - \mu_k^{-s} Q_k(s)||_{\rho_k} < \frac{1}{k^2}.$$

Let λ_{k+1} be an integer satisfying $\lambda_{k+1} > \mu_k + \deg(Q_k) \geq \mu_k > \lambda_k$ and $\limsup_{n\in\mathbb{N}} n|d_n| - \max\{l|d_l|; \lambda_{k+1} > l > \mu_k + \deg(Q_k)\} < \frac{1}{k^2}$. We set

$$F_k(s) = \sum_{n=\mu_k}^{\lambda_{k+1}-1} d_n n^{-s} - \mu_k^{-s} Q_k(s).$$

Since $||\lambda_k^{-s} P_k(s)||_{\sigma_k} < \frac{1}{k^2}$ and $||\mu_k^{-s} Q_k(s)||_{\sigma_k} < \frac{1}{k^2}$, the two Dirichlets series $\sum_{k\geq 1} \lambda_k^{-s} P_k(s)$ and $\sum_{k\geq 1} \mu_k^{-s} Q_k(s)$ are both in $\mathcal{D}_a(\mathbb{C}_+)$ (terms with index n appear only once, moreover the sequence $\tilde{\sigma}$ is decreasing, therefore the associed semi-norms increases). As the Dirichlet series f is absolutly convergent on $\mathbb{C}_+$, we obtain the same property for the following one (disjoint sums)

$$\sum_{k=1}^{\infty} \sum_{n=\mu_k}^{\lambda_{k+1}-1} d_n n^{-s}.$$

Therefore the Dirichlet series

$$\sum_{k\geq 1} \lambda_k^{-s} P_k(s) + \sum_{k=1}^{\infty} \sum_{n=\mu_k}^{\lambda_{k+1}-1} d_n n^{-s} - \sum_{k\geq 1} \mu_k^{-s} Q_k(s) = \sum_{k\geq 1} \left(\lambda_k^{-s} P_k(s) + F_k(s)\right)$$

is an element of $\mathcal{D}_a(\mathbb{C}_+)$. We denote this series by $\sum_{k\geq 1} a_n n^{-s}$. For $N = \lambda_k + \deg(P_k)$, we have

$$\sum_{n=1}^{N} a_n n^{-s} = \lambda_1^{-s} P_1(s) + F_1(s) + \cdots + \lambda_{k-1}^{-s} P_{k-1}(s) + F_{k-1}(s) + \lambda_k^{-s} P_k(s).$$

Similarly, we define a second Dirichlet series from $\mathcal{D}_a(\mathbb{C}_+)$

$$-\sum_{k\geq 1} \lambda_k^{-s} P_k(s) + \sum_{k=1}^{\infty} \sum_{n=\lambda_k}^{\mu_k-1} d_n n^{-s} + \sum_{k\geq 1} \mu_k^{-s} Q_k(s) = \sum_{k\geq 1} \left(R_k(s) + \mu_k^{-s} Q_k(s)\right),$$

and we denote this series $\sum_{n\geq 1} b_n n^{-s}$. By construction, we have the relation

$$\forall n \geq 1 \quad d_n = a_n + b_n.$$

Moreover by Corollary 3.5, we know that all the coefficients which appear in the decomposition of $\lambda_k^{-s}P_k(s) = \sum_{n\geq 1} p_{k,n}(\lambda_k n)^{-s}$ and $\mu_k^{-s}Q_k(s) = \sum_{n\geq 1} q_{k,n}(\mu_k n)^{-s}$ (denoted by $(r_{k,n})$ and $(s_{k,n})$ respectively) satisfy

$$\lambda_k n|r_{k,\lambda_k n}| = \lambda_k n|p_{k,n}| \leq \frac{1}{k^2}$$

and

$$\mu_k n|s_{k,\mu_k n}| = \mu_k n|q_{k,n}| \leq \frac{1}{k^2}.$$

Hence, the coefficients of the series $\sum_{k\geq 1} \lambda_k^{-s}P_k(s) - \sum_{k\geq 1} \mu_k^{-s}Q_k(s)$ (denoted by (t_n)) satisfy $n|t_n| \to 0$. Therefore, we have the following estimates

$$\limsup_{n\in\mathbb{N}} n|a_n| \leq \limsup_{n\in\mathbb{N}} n|d_n| \quad \text{and} \quad \limsup_{n\in\mathbb{N}} n|b_n| \leq \limsup_{n\in\mathbb{N}} n|d_n|.$$

In the second hand, for l satisfying $\mu_k + \deg(Q_k) < l < \lambda_{k+1}$, we have $d_l = a_l$ and

$$\limsup_{n\in\mathbb{N}} n|d_n| - \max\{l|d_l|; \mu_k + \deg(Q_k) < l < \lambda_{k+1}\} < \frac{1}{k^2}.$$

As an easy consequence, we have $\limsup_{n\in\mathbb{N}} n|d_n| = \limsup_{n\in\mathbb{N}} n|a_n|$. Similarly we have the second equality $\limsup_{n\in\mathbb{N}} n|d_n| = \limsup_{n\in\mathbb{N}} n|b_n|$. To conclude the proof, we have to prove that the two elements $\sum_{n\geq 1} a_n n^{-s}$ and $\sum_{n\geq 1} b_n n^{-s}$ are both in $\mathcal{W}_1$. Let K be an admissible compact set in $\mathbb{C}_-$ and h be a continuous function on K, analytic inside the interior of K. For every $\varepsilon > 0$ and $v \in \mathbb{N}$, we want to find $N \geq v$ such that

$$\sup_{s\in K}\left|h(s) - \sum_{n=1}^{N} a_n n^{-s}\right| < \varepsilon.$$

There exists a sequence f_λ $(\lambda = 1, 2, \ldots)$ such that

$$\sup_{s\in K} |h(s) - f_\lambda| < \frac{\varepsilon}{2}.$$

Moreover, there exists a sequence $(\rho_p)_{p\geq 0}$ such that $K \subset K_{\rho_p}$ and we can consider the set $\left\{(K_\rho, f_\lambda + q); q \in \mathbb{Q}; 0 < q < \frac{\varepsilon}{4}\right\}$ to conclude as in Proposition 5.5 [12] $\square$

Corollary 3.7 *We have the strict inclusion* $\mathcal{W}_a \subsetneq \mathcal{W}_1$.

Proof. It suffices to apply Theorem 3.6 with $d_n = \dfrac{1}{n}$. Corollary 2.3 implies that g_1 and g_2 cannot be in $\mathcal{W}_a$. $\square$

In the universal set $\mathcal{W}_1$, a natural problem is the existence of universal series which converge and moreover are continuous on the imaginary axis. In case of Taylor series this existence is proved by the study of universality on the Banach space $A(\mathbb{D})$ of analytic function on $\mathbb{D}$, continuous on the torus $\mathbb{T}$ [12]. In the next section we give also in the Dirichlet case a positive answer introducing universal series in the Wiener-Dirichlet algebra.

4 Universality in the Wiener-Dirichlet algebra.

4.1 Preliminary results.

The classical Wiener algebra of absolutely convergent Taylor series in one variable is the set of functions $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ such that $\sum_{n=0}^{+\infty} |a_n| < +\infty$.

Similarly, we can define Wiener-Dirichlet algebra, denoted in the following by $\mathcal{D}_w$. A Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is in this algebra if $||f|| = \sum_{n=1}^{+\infty} |a_n| < +\infty$. Endowed with this norm, $\mathcal{D}_w$ is obviously a Banach algebra. These two algebras are not completely similar. In effect, it is well-known that the spectrum of the Wiener algebra is $\overline{\mathbb{D}}$, the closed unit disk. For the Wiener-Dirichlet algebra, using the classical Bohr point of view [5], we can prove that its spectrum is $\overline{\mathbb{D}}^{\infty}$. At last, we can easily remark that this Wiener-Dirichlet algebra is a subset of $\mathcal{D}_a(\mathbb{C}_+)$.

Definition 4.1 *We denote by $\mathcal{U}_{wd}$ the set of all Dirichlet series $h \in \mathcal{D}_w$ satisfying : for every admissible compact set $K \subset \mathbb{C}_-$ and for every function g, continuous on K and analytic in $\overset{\circ}{K}$, there exists a sequence of integers $(\lambda_n)_{n\geq 0}$ such that we have*

$$\sup_{z\in K} |S_{\lambda_n}(h)(z) - g(z)| \underset{n\to+\infty}{\longrightarrow} 0.$$

After giving a version of Mergelyan's theorem, we give relations between the set $\mathcal{U}_{wd}$ and subsets of $\mathcal{D}_w$ realising the given estimations with Dirichlet polynomials. In the second hand, we prove that these subsets are opened and that their union is dense. We conclude using category type arguments. The methods are now classic to obtain analogous results (see [13], [6] or [8]) in the spaces of analytic functions.

As a consequence of main result, we obtain information on the universal set $\mathcal{W}_1$ defined below. Moreover, we precise the strict inclusion $\mathcal{W}_a \subsetneq \mathcal{W}_1$. So let us begin with a version of Mergelyan's Theorem for the Wiener-Dirichlet algebra.

Proposition 4.2 *Let K be admissible for Dirichlet series compact set included in $\mathbb{C}_-$. Let also g be a continuous function on K and analytic in the interior of K. Let also be $f \in \mathcal{D}_w$. Then, for every strictly positive number ε, there exists a Dirichlet polynomial $h(s) = \sum_{n\geq 1} h_n n^{-s}$ satisfying*

$$\begin{cases} \sup_{s\in K} |h(s) - g(s)| < \varepsilon \\ \\ ||h - f|| < \varepsilon. \end{cases}$$

Proof. We use the notations of Lemma 2 from [3] with the special case $f \in \mathcal{D}_w$. We write $K = K_1 \cup \cdots \cup K_d$, with $K_i \subset \{s \in \mathbb{C}; a_i \leq \Re(s) \leq b_i\}$, where $0 > b_1 \geq a_1 > b_2 \ldots b_d \geq a_d$ and $b_i - a_i < \frac{1}{2}$. We will use d-times the Bagchi's result. We approximate simultaneously $f(s) = \sum_{n=1}^{+\infty} a_n n^{-s}$, and g on K_1. Let

$\sigma_1 > 1$ such that $K_1 + \sigma_1 \subset \{s \in \mathbb{C}; \frac{1}{2} < \Re(s) < 1\}$. Let be $n_1 \in \mathbb{N}$ such that

$$\begin{cases} \displaystyle\sum_{n \geq n_1+1} \frac{1}{n^{\sigma_1}} < \frac{\varepsilon}{2}. \\ \displaystyle\sup_{s \in K_1} \left| f(s) - \sum_{n=1}^{n_1} a_n n^{-s} \right| < \frac{\varepsilon}{2}. \end{cases}$$

Then, we follow the construction of F. Bayart for $K_2 \cup \cdots \cup K_d$. Hence, we obtain a Dirichlet polynomial

$$h(s) = \sum_{n=1}^{n_1} a_n n^{-s} + \sum_{l=1}^{d} \sum_{n=n_l+1}^{m_l} b_n^{(l)} n^{-\sigma_l} n^{-s} = \sum_{n=1}^{\infty} h_n n^{-s}.$$

And, let us prove that h satifies the wanted inequalities. Indeed, we have

$$||h - f|| \leq \frac{\varepsilon}{2} + \sum_{l=1}^{d} \sum_{n=n_l+1}^{m_l} \frac{|b_n^{(l)}|}{n^{\sigma_l}} \leq \frac{\varepsilon}{2} + \sum_{n=n_1+1}^{+\infty} \frac{1}{n^{\sigma_1}} \leq \varepsilon.$$

The second inequelity follows from Lemma 2 of [3]. □

Remark 4.3 In this version of the Mergelyan's Theorem, we donot allow that the compact sets have a nonempty intersection with the imaginary axis. This restriction allow us to take $\sigma = 0$ in the version of F. Bayart. In effect, we need to have $\sigma_1 > 1$ to obtain a convergent series $\displaystyle\sum_{j=1}^{+\infty} j^{-\sigma_1}$ (see Lemma 3.3).

In the following, we use notations 3.2. We have easily the following lemma.

Lemma 4.4 *The family of Dirichlet polynomials $(f_j)_{j \in \mathbb{N}}$ is a dense set of $\mathcal{D}_w$ and for the topology of the uniform (on every compact set) convergence.*

Definition 4.5 *According to the preceeding definitions, for all positive integers ρ, j, n, s we define the sets $\mathcal{O}_w(\rho, j, s, n) \subset \mathcal{D}_w$ by*

$$\mathcal{O}_w(\rho, j, s, n) = \left\{ g \in \mathcal{D}_w \text{ such that } \sup_{z \in K_\rho} |S_n(g)(z) - f_j(z)| < \frac{1}{s} \right\}.$$

With these sets we have a complete representation of $\mathcal{U}_{wd}$.

Lemma 4.6 *We have the following equality*

$$\mathcal{U}_{wd} = \bigcap_{\rho=1}^{+\infty} \bigcap_{j=0}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcup_{n=1}^{+\infty} \mathcal{O}_w(\rho, j, s, n).$$

Proof. Let g be a Dirichlet series $\mathcal{D}_w$ of the righthand-side set. Let $K \subset \mathbb{C}_-$ be an admissible compact set for Dirichlet series and $\Phi : K \to \mathbb{C}$ be continuous function on K and analytic in $\overset{\circ}{K}$. For all $\varepsilon > 0$, we just have to find an integer $n_0 \in \mathbb{N}$ such that we have

$$\sup_{z \in K} |S_{n_0}(g)(z) - \Phi(z)| < \varepsilon.$$

Using Proposition 4.2, there exists a Dirichlet polynomial p satisfying

$$\sup_{z \in K} |p(z) - \Phi(z)| < \frac{\varepsilon}{2} \text{ and } ||p - h|| < \frac{\varepsilon}{2}.$$

The inequalities allow us to conclude. The inverse inclusion is obvious. □

4.2 Main results.

We prove first that each set $\bigcup_{n \geq 1} \mathcal{O}_w(\rho, j, s, n)$ is opened. Obviously, we just have to prove that $\mathcal{O}_w(\rho, j, s, n)$ is opened. Then we can conclude on the universality of the set $\mathcal{U}_{wd}$.

Proposition 4.7 *For all j, s, n and ρ positive integers, the subsets $\mathcal{O}_w(\rho, j, s, n)$ are opened in $\mathcal{D}_w$.*

Proof. We denote by M the lower boundary of the real parts of complex numbers included in K_ρ. Let $g(z) = \sum_{j \geq 1} g_j j^{-z}$ be a Dirichlet series in $\mathcal{O}_w(\rho, j, s, n)$, which means that we have

$$\sup_{z \in K_\rho} |S_n(g)(z) - f_j(z)| < \frac{1}{s}.$$

Let ε_1 be the following stricly positive real number :

$$\varepsilon_1 = n^M \left(\frac{1}{s} - \sup_{z \in K_\rho} |S_n(g)(z) - f_j(z)| \right) > 0.$$

Let $h = \sum_{j \geq 1} h_j j^{-z}$ be an element in $\mathcal{D}_w$ such that $||h - g|| \leq \varepsilon_1$. Now, we can overestimate $|S_n(h)(z) - f_j(z)|$. One has for $z \in K_\rho$

$$\begin{aligned} |S_n(h)(z) - f_j(z)| &\leq |S_n(h-g)(z)| + |S_n(g)(z) - f_j(z)| \\ &\leq |S_n(h-g)(z)| + \sup_{z \in K_\rho} |S_n(g)(z) - f_j(z)| \end{aligned}$$

Afterwards, if we denote $S_n(h-g)(z) = \sum_{j=1}^{n} (h_j - g_j) j^{-z}$, we have

$$|S_n(h-g)(z)| \leq \sum_{j=1}^{n} |h_j - g_j| n^{-M} \leq n^{-M} ||h - g||.$$

Consequently, one has for $z \in K_\rho$

$$\begin{aligned} |S_n(h)(z) - f_j(z)| &\leq n^{-M} ||h - g|| + \sup_{z \in K_\rho} |S_n(g)(z) - f_j(z)| \\ &\leq n^{-M} \varepsilon_1 + \sup_{z \in K_\rho} |S_n(g)(z) - f_j(z)| < \frac{1}{s}. \end{aligned}$$

Therefore, the set $\mathcal{O}_w(\rho, j, s, n)$ is open. □

Theorem 4.8 *With previous notations, for all j, ρ and s integers, the sets $\bigcup_{n\geq 0} \mathcal{O}_w(\rho, j, s, n)$ are dense subsets $\mathcal{D}_w$.*

Proof. Let $d \in \mathcal{D}_w$ be written $d(z) = \sum_{j=1}^{+\infty} a_j j^{-z}$. We want to obtain a sequence of Dirichlet polynomials which converges to d. Let be $\varepsilon \in]0; \frac{1}{2s}[$. We denote by $d_r(z)$ the partial sum $\sum_{j=1}^{r} a_j j^{-z}$ and we choose r such that $||d - d_r|| < \varepsilon$. Now, we approximate the Dirichlet polynomial d_r with an element in $\bigcup_{n=0}^{\infty} \mathcal{O}_w(\rho, j, s, n)$, with an index j in $\mathbb{N}$. According to Proposition 4.2, for all $m > 0$, there exists a Dirichlet polynomial p_m satisfying the following inequalities

$$\sup_{z \in K_\rho} |p_m(z) - f_j(z)| < \frac{1}{m} \text{ and } ||p_m - d_r|| < \frac{1}{m}.$$

For all $m \geq s$, the Dirichlet polynomials are in $\bigcup_{n=0}^{\infty} \mathcal{O}_w(\rho, j, s, n)$ and we have

$$||p_m - d|| \leq \varepsilon + \frac{1}{m}.$$

The conclusion follows. $\square$

Theorem 4.9 *The set $\mathcal{U}_{wd}$ is a G_δ-dense set included in $\mathcal{D}_w$.*

Proof. Lemma 4.6 implies that $\mathcal{U}_{wd}$ is a denombrable intersection of dense opened sets of $\mathcal{D}_w$. Hence, the result is a direct consequence of the Baire's theorem. $\square$

As a straightfoward consequence of the preceeding theorem, we obtain the following explicit result of approximation.

Corollary 4.10 *Let f be in $\mathcal{D}_w$. Then, for all $\varepsilon > 0$ and ψ, continuous on K (admissible compact set of $\mathbb{C}_-$) and analytic in $\overset{\circ}{K}$, there exists a sequence of integers $(\lambda_n)_{n\geq 0}$ and a Dirichlet series $h \in \mathcal{D}_w$ satisfying*

$$\begin{cases} ||h - f|| < \varepsilon \\ \sup_{z \in K} |S_{\lambda_n}(h)(z) - \psi(z)| \underset{n \to +\infty}{\longrightarrow} 0 \end{cases}$$

Remark 4.11 The universal set $\mathcal{U}_{wd}$ is dense in $\mathcal{D}_w$. Obviously, the property is also true in $\mathcal{D}_a(\mathbb{C}_+)$. Every Dirichlet series of $\mathcal{U}_{wd}$ converges on the imaginary axis and is continuous on this set. Moreover we have obviously $\mathcal{U}_{wd} \subset \mathcal{W}_1$.

Corollary 4.12 *We have the inclusion $\mathcal{W}_a \subset (\mathcal{W}_1 \cap \mathcal{U}_{wd}^c)$.*

Proof. Every function from $\mathcal{W}_a$ converges nowhere on the imaginary axis. Therefore we have $\mathcal{W}_a \cap \mathcal{U}_{wd} = \emptyset$. $\square$

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